Modifications of Lusin's example of Σ^1_1-complete set

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Lusin's example

Theorem (Lusin)

Set

$$L = \{x \in (\omega \setminus \{0\})^\omega : (\exists k_0 < k_1 < k_2 < \ldots)(\forall i)(x(k_i) | x(k_{i+1}))\}$$

is $\mathbf{\Sigma}^1_1$*-complete.*

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Now let X be any countable set and $\leqslant_X$ an ordering of X. Define:

$$L_X = \{y \in X^\omega : (\exists k_0 < k_1 < k_2 < \ldots)(\forall i)(y(k_i) \leqslant_X y(k_{i+1}))\}.$$

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Fact

For every $(X, \leqslant_X)$ *set* L_X *is analytic.*

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If $X = \mathbb{Z}$ and $\leqslant_X = \leqslant$, then L_X is a Borel set.

$$L_{\mathbb{Z}} = L_= \cup \{y \in \mathbb{Z}^\omega : (\forall n \in \mathbb{Z})(\exists k \in \omega)(y(k) > n)\}$$

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Fact

If $(X, \leqslant_X)$ is a well-ordering, then $L_X = X^\omega$.

Result for $\mathbb{Q}$

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We will construct a Borel function $f : Trees \to \mathbb{Q}^\omega$ such that $f^{-1}(L_X) = IF$. Let σ_n be enumeration of $\omega^{<\omega}$ satisfying

$$\sigma_n \subseteq \sigma_m \implies n < m$$

and $\varphi : \omega^{<\omega} \to \mathbb{Q}$:

$$\varphi(x) = \left(0.\underbrace{000\ldots0}_{x_0}1\underbrace{000\ldots0}_{x_1}1\underbrace{000\ldots0}_{x_2}1\ldots1\right)_2 .$$

Now define

$$f(T)(n) = \begin{cases} \varphi(\sigma_n), & \sigma_n \in T \\ -n, & \sigma_n \notin T \end{cases} .$$

Characterization of linear orders

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Let $X \subseteq \mathbb{Q} \cap [0,1]$, $\leqslant_X = \leqslant$. *Then*

- *if* $|\overline{X}| = \omega$, L_X *is Borel,*
- *if* $|\overline{X}| = \mathfrak{c}$, X *contains a* $\leqslant$*-dense subset.*

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- when X is dense-in-itself, $\overline{X}$ is NWD perfect set, from which we can construct $\leqslant$-dense set,
- when X is scattered, $\overline{X}$ contains a NWD perfect C. Furthermore it can be shown that

$$\overline{X \backslash C} \supseteq C.$$

Corollary

Let $(X, \leqslant_X)$ be a linear order. Then L_X is $\boldsymbol{\Sigma}^1_1$-complete if and only if X contains a $\leqslant_X$ $-$dense subset.

Question

What about partial orderings?

Thank you for attention.